GRAPH THEORY – EXAM

10 JANUARY 2022

Please note: This exam consists of 5 problems, worth 8p each. In order to pass this part of the course examination, you will need to obtain at least 18 points out of 40.

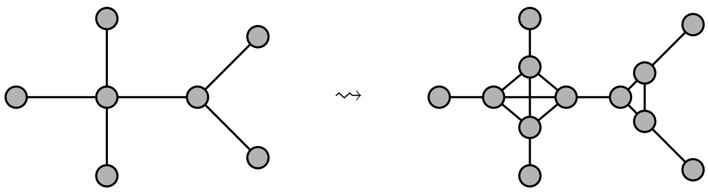
For solving this exam, no aids are allowed. When using colours, please abstain from using the colour red.

– *Good luck!*

Problem 1. (a) Define the notions of *tree* and *spanning tree.* (2p)

(b) Prove Cayley's formula, stating that a labelled K_n has n^{n-2} spanning trees. (4p)

(c) Consider a tree $T = (V, E)$. Construct the blow-up $B(T)$ of T as follows: Replace every $v \in V$ by a clique C_v on $\deg(v)$-many vertices. For an edge $\{v, w\}$ in T, draw an edge between one vertex of C_v and one vertex of C_w, such that no vertex v is adjacent to two vertices from cliques other than C_v. As an example, consider the following tree:

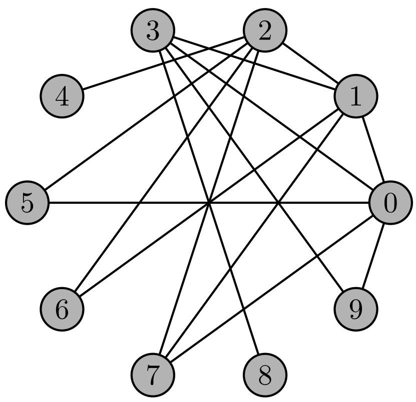


Show that

$$\prod_{v \in V} \deg(v)^{\deg(v)-2}$$

is the number of spanning trees in a labelled $B(T)$. (2p)

Problem 2. Consider the following weighted graph:



where the edge-weight of an edge $\{i, j\}$ with $i < j$ is given by $1 + j - 2^i$.

(a) Employ Prim's or Kruskal's algorithm to find a minimum spanning tree. Your solution should contain all relevant steps of the algorithm. (4p)
(b) Use Dijkstra's algorithm to determine the distance between vertices 0 and 7. (*Hint:* You do not need to consider the vertices 4,8,9 for this). (4p)

Problem 3. (a) State and prove Euler's formula, relating the number of vertices, edges, and faces of a planar connected graph. (4p)

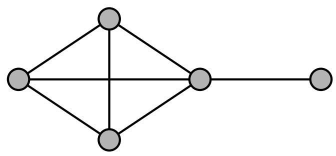
One consequence of Euler's formula is that $|E| \leq 3|V| - 6$ holds for a planar graph $G = (V, E)$, as was shown in the lectures.

(b) Prove that a planar graph has a vertex of degree ≤ 5. (1p)
(c) Show that a planar graph is 6-colourable (you may not use the 4- or 5-colour theorem for this). (3p)

Problem 4. By C_n^2 we denote the cycle graph C_n where additional edges have been drawn between all pairs of vertices which have distance 2 in C_n. As in the lectures, $\chi(\cdot)$ denotes the chromatic number.

(a) Show that C_{2n}^2 is planar for $n \geq 2$. (2p)
(b) Show that C_{2n+1}^2 is non-planar for $n \geq 2$. (2p)
(c) Show that $\chi(C_{2n}^2) = 3$ whenever $n \geq 2$ is a multiple of 3. (2p)
(d) Show that $\chi(C_{2n}^2) = 4$ whenever $n \geq 2$ is not a multiple of 3. (2p)

Problem 5. Let H denote the following graph:



(a) How is the random graph $G(n, p)$ constructed? (2p)
(b) For $p = p(n) = n^{-7/10}$, show that the probability of $G(n, p)$ containing a K_4 converges to 0 as $n \to \infty$. (*Hint:* Recall the first moment method, stating that $\mathbf{P}[X = 0] \leq \mathbf{E}[X]$ for a non-negative, integer-valued random variable X.) (2p)
(c) For $p = p(n) = n^{-7/10}$, show that the expected number of copies of H in $G(n, p)$ goes to infinity as $n \to \infty$. (3p)
(d) Despite the result of (c), show that for $p = p(n) = n^{-7/10}$ the probability of $G(n, p)$ containing a copy of H converges to 0 as $n \to \infty$. (1p)